On the Smoothed Complexity of Combinatorial Local Search

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Context

Polynomial Local Search (Johnson et al. 1988)

- Finding a sink in a DAG
- Neighbourhood can be queried in **poly** time, while graph can be **exponential**.
- PLS-complete problems:
  - Local Max-Cut/Flip
  - MaxSAT/Flip
  - TSP/k-Opt
  - Potential Games/Best-Responses

Smoothed Analysis (Spielman, Teng 2004)

- Bridging the gap between worst-case and average-case analysis
- Smoothed Complexity in terms of **expected number of steps**
- **Parameterized** noise
- Variance of Gaussian perturbation
- Input given by probability density functions, bounded by ϕ

Congestion Games (Rosenthal 1973)

- n Players and resource set $\mathcal{R}$ with latency function κ_r for each $r \in \mathcal{R}$
- Strategy set $\Sigma_i \subseteq \mathcal{R}$ for every player
- Strategy profile $\boldsymbol{\sigma} \in \prod_{i=1}^{n} \Sigma_i$
- Cost of player depends on their used resources and their load ℓ_r.
- Rosenthal potential function Φ serves as local search objective for PLS.

Combinatorial Local Optimization

Our model for PLS problems, called combinatorial local optimization (CLO):

- $S \subseteq \{0, 1, \ldots, M\}^{\nu} \times \{0,1\}^{\overline{\nu}}$ as set of configurations, consisting of **cost and non-cost part**
- Cost vector $\boldsymbol{c} = (c_1, \ldots, c_\nu) \in [-1,1]^{\nu}$
- **Linear** cost $C(\boldsymbol{s}) = \sum_{i=1}^{\nu} c_i s_i$ for $\boldsymbol{s} \in S$
- Smoothness: independently draw each c_i from densities
  $$f_i: [-1,1] \to [0, \phi].$$

Congestion Games as CLO problem

- Explicit cost model: Values of κ_r are given as lists $(\kappa_r(1), \ldots \kappa_r(n))$.
- Smoothness model: independent perturbations in each value
- Potential function is linear:
  $$\Phi(\boldsymbol{\sigma}) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n} \kappa_r(i)[i \leq \ell_r(\boldsymbol{\sigma})]$$
- Parameters $M = 1, \nu = n \cdot m$

Main Theorem

$(\boldsymbol{\lambda}, \boldsymbol{\beta}, \boldsymbol{\mu})$**-separability**:

- λ sets of *similar* transitions cover the neighbourhood graph.
- Each set affects at most β different (*clusters* of) coordinates.
- At most μ different *configuration differences* appear in each cluster.

Main Theorem: On any (λ, β, μ)-separable smoothed CLO instance, standard local search terminates after at most
$$3 \cdot \mu^{\beta} \lambda \cdot \nu^2 \cdot \phi \cdot\ M \log(M+1)$$
many steps in expectation.

Showcase: Congestion Games

Consider all best-response transitions where player i deviates from a_i to a_i':
We can represent the configurations $\boldsymbol{s}(\boldsymbol{\sigma}), \boldsymbol{s}(\boldsymbol{\sigma}')$ in tabular form:

$\mathcal{R} \setminus \ell_r$	1	2	3	4	5	6
r_1	1	1	1	1	1	0
r_2	1	1	0	0	0	0
r_3	1	0	0	0	0	0

$\boldsymbol{s}(\boldsymbol{\sigma})$

$r_1 \in a_i \setminus a_i'$ →
$r_2 \in a_i' \setminus a_i$ →
$r_3 \notin a_i \,\Delta\, a_i'$ →

$\mathcal{R} \setminus \ell_r$	1	2	3	4	5	6
r_1	1	1	1	1	0	0
r_2	1	1	1	0	0	0
r_3	1	0	0	0	0	0

$\boldsymbol{s}(\boldsymbol{\sigma}')$

Consider each row as a *coordinate cluster*:

- For $r \in\ a_i \,\Delta\, a_i'$, the difference in the configuration is some standard basis vector.
- For $r \notin\ a_i \,\Delta\, a_i'$, the configuration doesn't change.
  $\Rightarrow\ \mu\ =\ n, \beta\ = \max |a \,\Delta\, a'|$
- In total, there exist $\lambda\ =\ n \cdot k(k-1)$ deviation sets.
- By application of the main theorem, the smoothed running time is bounded by
  $$\mathcal{O}(n^{B+3} k^2 m^2 \phi)$$
  for $B = \max |a \,\Delta\, a'|$. Congestion games remain PLS-complete for constant B.

Our Applications

Problem	Perturbation	PLS-hardness	Smoothed Complexity	Novelty
(A)TSP/k-Opt	Edge/arc weights	For constant k	$\mathcal{O}(4^{k^2} m^{k+2} \phi)$	Unified proof for any k
MaxSAT/k-Flip	Clause weights	Even with bounded variable occurrence (B)	$\mathcal{O}(3^{kB} n^k m^2 \phi)$	Parameterization by variable occurrence
Local Max-k-Cut	Edge weights	Even for constant degree graphs	$\mathcal{O}(3^{\Delta(G)} n m^2 \phi)$	Unified parameterization by degree and arbitrary k
Network coordination games	Payoffs	Even for constant degree graphs	$\mathcal{O}(k^{4\Delta(G)+6} n m^2 \phi)$	Parameterization by degree, not exponential in strategy size
Congestion games	Latency functions: explicit, polynomial, and step-function	Even for constantly restrained games	$\mathcal{O}(n^{B+3} k^2 m^2 \phi)$ explicit B-restrained games	Smoothness model and analysis
Network congestion games	Same as above	Even for constantly compact games	Polynomial for (A, B)-compact games with constant B	Smoothness model and analysis
Weighted set problems (3D-Matching, Set-Cover, Hitting-Set), Maximum Constraint Assignment	Weights	Constant neighbourhood sizes	Polynomial for any constant neighbourhood size	Smoothness model and analysis