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Problem Set 1

Applying Results on the Approximability of the Set Cover Problem

Approximation Algorithms (MA5517)

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This problem set will be discussed in the tutorial on October 30th/31st, 2018.

Problem 1.1 (Feedback Vertex Set Problem in Tournaments)

In the directed version of the FeedbackVertexSet problem, a directed graph $G = (V, A)$ is given. The task is to find a set of vertices $W \subseteq V$ of minimum cardinality such that the graph $G - W$ which results from G by deleting all vertices in W is acyclic.

i) Show that if G is a tournament[a] then it is acyclic if and only if it contains no cycle of length three.

ii) Find a 3-approximation algorithm for FeedbackVertexSet when the given graph is a tournament.

Problem 1.2 (Uncapacitated Facility Location Problem)

In the UncapacitatedFacilityLocation problem, we have a set of facilities F and a set of clients C. There is a cost $f_i \geq 0$ of opening a facility $i \in F$. Further for each facility $i \in F$ and client $j \in C$, there is a cost $c_{ij} \geq 0$ of assigning client j to an opened facility i. The goal is to open a subset of facilities $F' \subseteq F$ so as to minimize the total cost of opening facilities and assigning clients to opened facilities. In other words, we wish to find $F' \subseteq F$ which minimizes $\sum_{i \in F'} f_i + \sum_{j \in C} \left(\min_{i \in F'} c_{ij} \right)$.

i) Show that there is $\alpha > 0$ such that no $(\alpha \ln |C|)$-approximation algorithm exists for UncapacitatedFacilityLocation, unless P = NP.

ii)* Give an $\mathcal{O}(\ln |C|)$-approximation algorithm for UncapacitatedFacilityLocation.

Problem 1.3 (Weighted Vertex Cover Problem in Planar Graphs)

In the WeightedVertexCover problem, an undirected graph $G = (V, E)$ with weights $w_v \geq 0$ for all vertices $v \in V$ is given. The task is to find a set of vertices $W \subseteq V$ of minimum total weight such that W intersects every edge in E.

i) Prove that the set of extreme points of the polyhedron given by the inequalities

$$\begin{aligned} x_u + x_v &\geq 1 \qquad \text{for all } \{u, v\} \in E \\ x_v &\geq 0 \qquad \text{for all } v \in V \end{aligned}$$

is a subset of $\{0, 1/2, 1\}^V$.

ii) Give a $3/2$-approximation algorithm for WeightedVertexCover when the input graph is planar. You may assume that polynomial-time linear program solvers return extreme points, and that there is a polynomial-time algorithm to 4-color[b] any planar graph.

[a] A tournament is a directed graph $G = (V, A)$ such that for every pair of distinct vertices $u, v \in V$ either $(u, v) \in A$ or $(v, u) \in A$.

[b] A k-coloring of a graph is an assigment of one of k colors to every vertex such that any pair of adjacent vertices is assigned different colors.

Problem 1.4 (Metric Asymmetric Traveling Salesman Problem)

In the MetricAsymmetricTravelingSalesman problem, we are given as input a complete directed graph $G = (V, A)$ with a length $\ell_{vw} \geq 0$ for each arc $(v, w) \in A$, such that the arc lengths obey the triangle inequality, i.e. $\ell_{uw} \leq \ell_{uv} + \ell_{vw}$ holds for all $u, v, w \in V$. The goal is to find a tour of minimum length, i.e., a directed cycle that contains each vertex exactly once, such that the sum of the lengths of the arcs in the cycle is minimized.

One approach to finding an approximation algorithm for this problem is to find a minimum-length strongly connected[c] Eulerian[d] subgraph of the input graph and shortcut it to a tour. To find a strongly connected Eulerian subgraph consider the following procedure.
First, find a minimum mean-length cycle[e] in the graph. Then, choose one vertex of the cycle arbitrarily, remove all other vertices of the cycle from the graph, and repeat until only one vertex of the graph is left. Return the subgraph consisting of all the arcs from all the cycles found.

i) Prove that the subgraph found by the procedure is a strongly connected Eulerian subgraph of the input graph.

ii)* Show that this algorithm represents a $2H_{|V|}$-approximation algorithm for MetricAsymmetricTravelingSalesman, where H_n denotes the n-th harmonic number.

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[c] A directed graph is strongly connected if for any pair of vertices $v, w \in V$ there is a directed v-w-path and w-v-path.
[d] A directed graph is Eulerian if the in-degree of each vertex equals its out-degree.
[e] A minimum mean-length cycle is a directed cycle that minimizes the ratio of the total length of the arcs in the cycle to the number of arcs in the cycle. You may use that such a cycle can be found in polynomial time.

Grade Bonus:

A bonus for the final exam can be achieved by continuous participation in the tutorial classes. A solution to each subproblem is presented by a student who indicated in the beginning of the class to be able to do so *comprehensibly.* Each student who is prepared for the presentation of at least 2/3 *of the total number of subproblems on all problem sets* is granted a bonus for the grade of their final exam which improves a *passed* exam by one grade (e.g. 4.0 becomes 3.7, 2.7 becomes 2.3, etc. Note that 1.0 cannot be improved). The bonus applies to both, the first and retake attempt of the final exam. Fraud, in particular, if a student is not able to present a solution as indicated, may lead to their exclusion from the bonus system. (Sub)problems marked with an asterisk (*) are *supplementary* and not taken into account for the total number of subproblems. You will, however, be credited for preparing them.