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# Problem Set 4

## Linear Programming Relaxations and Rounding

Approximation Algorithms (MA5517)

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This problem set will be discussed in the tutorials on November 20th/21st, 2018.

**Problem 4.1** (Weighted Sum of Completion Times with Precedence Constraints)

We consider a scheduling problem on a single machine. There are $n$ jobs with processing times $p_j \geq 0$ and non-negative weights $w_j \geq 0$ for all $j \in [n]$. In addition, we are given a partial order $\preceq$ on the jobs. For a schedule, which is an ordering of the jobs, the completion time $C_j \geq 0$ of job $j \in [n]$ is the point in time when $j$ is completed. The goal is to find a total ordering of the jobs that extends $\preceq$ and minimizes the weighted sum of completion time $\sum_{j\in[n]} w_j C_j$.

i) Smith's rule states to process the jobs in non-increasing order w.r.t. the ratio of their weight to their processing time. Prove that Smith's rule gives an optimal schedule for the problem without precedence constraints.

ii) Show that the following linear program is a relaxation of the problem with precedence constraints.

$$\begin{aligned}
\min \quad & \sum_{j\in[n]} w_j C_j \\
\text{s.t.} \quad & \sum_{j\in J} p_j C_j \geq \tfrac{1}{2}\sum_{j\in J} p_j^2 + \tfrac{1}{2}\Big(\sum_{j\in J} p_j\Big)^2 && \text{for all } J \subseteq [n] \\
& C_j - C_i \geq p_j && \text{for all } i,j \in [n] : i \prec j \\
& C \in \mathbb{R}^n_{\geq 0}
\end{aligned}$$

*Hint: Use (i).*

iii) Let $C^* \in \mathbb{R}^n_{\geq 0}$ be an optimal solution to the LP-relaxation. Consider the schedule defined by ordering the jobs in non-decreasing order w.r.t. $C^*_j$. Prove that this schedule is a 2-approximation.

iv)* Deduce a 2-approximation algorithm for minimizing the weighted sum of completion times for scheduling on a single machine with precedence constraints. You may use that submodular[a] functions can be minimized in polynomial time.

**Problem 4.2** (Pipage Rounding)

In the MaximumCut problem, we are given as input an undirected graph $G = (V, E)$ with non-negative weights $w_e \geq 0$ for all $e \in E$. The goal is to find a set of vertices $W$ so as to maximize the weight of the edges that leave $W$. We consider a variant of the problem in which the size of $W$ has to agree with an integer $k \leq |V|/2$ that is part of the input.

i) Show that the variant of MaximumCut is modeled by the following integer program with non-linear objective function based on $f : [0,1]^2 \to \mathbb{R},\ (y,z) \mapsto y + z - 2yz$.

$$\begin{aligned}
\max \quad & \sum_{\{u,v\}\in E} w_{\{u,v\}} f(x_u, x_v) \\
\text{s.t.} \quad & \mathbb{1}^\top x = k \\
& x \in \{0,1\}^V
\end{aligned}$$

[a] For a finite set $E$, a function $f : 2^E \to \mathbb{R}$ is called submodular if $f(A \cap B) + f(A \cup B) \leq f(A) + f(B)$ for all $A, B \subseteq E$.

ii) Show that the following program with objective function based on $g : [0,1]^2 \to \mathbb{R},\ (y,z) \mapsto \min(y+z, 2-y-z)$ represents a relaxation of the above. Further, prove that $f(y,z) \geq \frac{1}{2}g(y,z)$ holds for any $y, z \in [0,1]$.

$$\begin{aligned} \max \quad & \sum_{\{u,v\}\in E} w_{\{u,v\}} g(x_u, x_v) \\ \text{s.t.} \quad & \mathbb{1}^\top x = k \\ & x \in [0,1]^V \end{aligned}$$

iii) Let $x$ be a solution to the relaxation. Argue that, for two fractional elements in $x$, it is possible to increase one by $\varepsilon \neq 0$ and decrease the other by $\varepsilon$ such that $\sum_{\{u,v\}\in E} w_{\{u,v\}} f(x_u, x_v)$ does not decrease and one of the two variables becomes integer.

iv) Use the arguments above to devise a $1/2$-approximation algorithm for the given variant of MaximumCut problem.

**Problem 4.3** (Maximum Directed Cut Problem)

In the MaximumDirectedCut problem, we are given as input a directed graph $G = (V, A)$. Each arc $a \in A$ has non-negative weight $w_a \geq 0$. The goal is to partition $V$ into two sets $W$ and $V \setminus W$ so as to maximize the total weight of the arcs going from $W$ to $V \setminus W$.

i) Show that the following integer program models the maximum directed cut problem and linearize it, i.e., find an equivalent (mixed-)integer linear program.

$$\begin{aligned} \max \quad & \sum_{(u,v)\in A} w_{(u,v)} \min(x_u, 1-x_v) \\ \text{s.t.} \quad & x \in \{0,1\}^V \end{aligned}$$

ii) Consider a randomized rounding algorithm for MaximumDirectedCut that solves a linear programming relaxation of the mixed-integer program and puts vertex $v \in V$ in $W$ with probability $1/4 + x_v/2$. Show that this gives a randomized $1/2$-approximation algorithm for MaximumDirectedCut.