Problem Set 6

The Primal-Dual Method and Iterated Rounding

Approximation Algorithms (MA5517)

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This problem set will be discussed in the tutorials on December 4th/5th, 2018.

Problem 6.1 (Multicut Problem in Trees)
Consider the MULTICUT problem in trees. In this problem, we are given an undirected tree $T = (V, E)$, cost $c_e \geq 0$ for each edge $e \in E$, and $k \in \mathbb{N}$ pairs of vertices $(s_i, t_i) \in V \times V, i \in [k]$. The goal is to find a set of edges $F \subseteq E$ of minimum cost such that for all $i \in [k]$, s_i and t_i are in different connected components of $(V, E \setminus F)$.

i) For $i \in [k]$, let P_i be the unique s_i-t_i-path in T. Show that the MULTICUT problem in trees can be formulated as the following integer program. Find the dual of its linear programming relaxation.

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{s.t.} \quad & \sum_{e \in P_i} x_e \geq 1 && \text{for all } i \in [k] \\ & x_e \in \{0, 1\} && \text{for all } e \in E \end{aligned}$$

ii) Suppose we root the tree at an arbitrary vertex $r \in V$. For $v \in V$, let $\operatorname{depth}(v)$ be the number of edges on the r–v-path in T. Further, let $\operatorname{lca}(s_i, t_i)$ be the lowest common ancestor of s_i and t_i, i.e., the vertex $v \in V$ on P_i whose depth is minimal.

Develop a primal-dual algorithm. In each iteration, increase the dual variable that corresponds to a violated constraint maximizing $\operatorname{depth}\big(\operatorname{lca}(s_i, t_i)\big)$. Prove that your algorithm returns a feasible solution after a polynomial number of steps.

iii) Prove that your algorithm gives a 2-approximation algorithm for the MULTICUT problem in trees.

Hint: Clean up in reverse order.

Problem 6.2 (Survivable Network Design Problem)
Let $G = (V, E)$ be an undirected graph and $w_e \geq 0$ be weights on the edge e for all $e \in E$.

For a weakly supermodular[a] function $f : 2^V \to \mathbb{N}$, consider the linear program

$$\begin{aligned} \min \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \geq f(S) && \text{for all } \emptyset \neq S \subsetneq V \\ & 0 \leq x_e \leq 1 && \text{for all } e \in E \end{aligned}$$

Let x be any basic feasible solution to the linear program such that $0 < x_e < 1$ holds for all $e \in E$. Denote the family of sets for which x fulfills the corresponding inequalities with equality by $\mathcal{S} := \big\{\emptyset \neq S \subsetneq V : \mathbb{1}_{\delta(S)}^\top x = f(S)\big\}$.

[a]For a finite set E, a function $f : 2^E \to \mathbb{R}$ is called weakly supermodular if $f(A) + f(B) \leq f(A \cap B) + f(A \cup B)$ or $f(A) + f(B) \leq f(A \setminus B) + f(B \setminus A)$ holds for all $A, B \subseteq E$.

i) Show that, for all $\emptyset \neq A, B \subsetneq V$, we have

$$\mathbb{1}_{\delta(A\cap B)} + \mathbb{1}_{\delta(A\cup B)} \leq \mathbb{1}_{\delta(A)} + \mathbb{1}_{\delta(B)} \qquad \text{and} \qquad \mathbb{1}_{\delta(A\setminus B)} + \mathbb{1}_{\delta(B\setminus A)} \leq \mathbb{1}_{\delta(A)} + \mathbb{1}_{\delta(B)}.$$

ii) Prove that if $A, B \in \mathcal{S}$, then

$$A \cap B, A \cup B \in \mathcal{S}, \text{ and } \delta(A \setminus B) \cap \delta(B \setminus A) = \emptyset, \qquad \text{or} \qquad A \setminus B, B \setminus A \in \mathcal{S}, \text{ and } \delta(A \cap B) \cap \delta(A \cup B) = \emptyset.$$

iii) Show that there is $\mathcal{L} \subseteq \mathcal{S}$ such that $\mathcal{L}$ is laminar[b] and $\{\mathbb{1}_{\delta(S)} : S \in \mathcal{L}\}$ is a basis of $\mathbb{R}^E$.

Hint: Try to extend a family $\mathcal{L}' \subseteq \mathcal{S}$ for which $\{\mathbb{1}_{\delta(S)} : S \in \mathcal{L}'\}$ does not span $\mathbb{R}^E$ by a set $S \in \mathcal{S}$ which minimizes the value $|\{T \in \mathcal{L}' :\ S \cap T \neq \emptyset \ \wedge\ T \setminus S \neq \emptyset \ \wedge\ S \setminus T \neq \emptyset\}|$.

[b]For a ground set E, a family of subsets $\mathcal{S} \subseteq 2^E$ is laminar if $A \subseteq B$, $A \supseteq B$, or $A \cap B = \emptyset$ holds for all $A, B \in \mathcal{S}$.