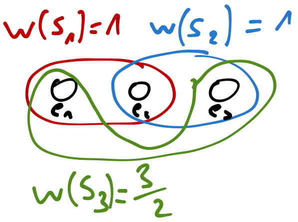
Example



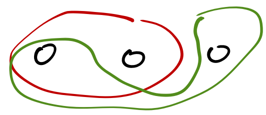
LP rounding

$$x(S_1)=x(S_2)=x(S_3)=\frac{1}{2}$$

$$\mathcal{F}'=\{S_1, S_2, S_3\}$$

$$\begin{array}{ccccc} Z^* & \leq & OPT & \leq & ALG \\ \| & & \| & & \| \\ \frac{1}{2}+\frac{1}{2}+\frac{3}{4}=\frac{7}{4} & & 2 & & \frac{7}{2} \end{array}$$

Primal-dual



$y(e_1)=1 \longrightarrow S_1$ tight

$y(e_3)=\frac{1}{2} \longrightarrow S_3$ tight

$$\begin{array}{ccccc} \sum_{e\in E} y(e) & \leq & OPT & \leq & ALG \\ \| & & \| & & \| \\ \frac{3}{2} & & 2 & & \frac{5}{2} \end{array}$$

Analysis of primal-dual

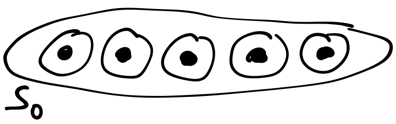
Proof of Theorem 1.2 $\mathcal{F}'$ is a set cover, y is a feasible dual solution.

$$\sum_{S\in\mathcal{F}'} w(S) \underset{\substack{\uparrow \\ \text{every } S\in\mathcal{F}' \text{ is tight} \\ \text{w.r.t. } y}}{=} \sum_{S\in\mathcal{F}'}\sum_{e\in S} y(e) = \sum_{e\in E}\sum_{S\in\mathcal{F}': e\in S} y(e) \leq \sum_{e\in E} f\cdot y(e) \underset{\substack{\uparrow \\ \text{LP duality}}}{\leq} f\, Z^* \quad \square$$

Example for greedy algorithm:

$E=\{e_1,\dots,e_n\}$, $\mathcal{F}=\{S_0, S_1,\dots,S_n\}$

$S_0=E$, $w(S_0)=1+\varepsilon$, $\varepsilon>0$



$S_i=\{e_i\}$, $w(S_i)=\frac{1}{i}$ for $i\in\{1,\dots,n\}$

In iteration i, Greedy takes set S_{n-i+1}. At beginning of iteration i, $n-i+1$ elements are not yet covered. S_0: $\frac{1+\varepsilon}{n-i+1}$ vs S_{n-i+1}: $\frac{1}{n-i+1}$.

$OPT = 1+\varepsilon$ greedy algorithm: H_n

(this shows that analysis is tight)

Analysis of the greedy algorithm

[discussed on Oct 24]

Proof of Theorem 1.3

ℓ iterations, $n_{\ell+1} = 0$

$$\sum_{i=1}^{\ell} w(S_i) \leq \sum_{i=1}^{\ell} \frac{n_i - n_{i+1}}{n_i} \mathrm{OPT} \leq \sum_{i=1}^{\ell} \left(\frac{1}{n_i} + \frac{1}{n_i - 1} + \ldots + \frac{1}{n_{i+1}+1} \right) \cdot \mathrm{OPT}$$

$$= \sum_{i=1}^{n} \frac{1}{i} \mathrm{OPT} = H_n \cdot \mathrm{OPT} \left(= O(\ln(n) \cdot \mathrm{OPT}) \right) \quad \square$$

Proof of Lemma 1.4

Consider iteration i: For every $S \in \mathcal{F}$ define $\hat{S} := S \setminus \bigcup_{j=1}^{i-1} S_j$.

Let $\mathcal{F}^*$ be an optimal solution to the SET COVER instance.

Then:

$$\frac{w(S_i)}{n_i - n_{i+1}} \cdot n_i \leq \frac{w(S_i)}{|\hat{S}_i|} \sum_{S \in \mathcal{F}^*} |\hat{S}| \leq \sum_{\substack{S \in \mathcal{F}^*: \\ \hat{S} \neq \emptyset}} |\hat{S}| \frac{w(S)}{|\hat{S}|} \leq \sum_{S \in \mathcal{F}^*} w(S) \quad \square$$

$|\hat{S}_i| = n_i - n_{i+1}$

$|\bigcup_{S \in \mathcal{F}^*} S| = n \geq n_i$

greedy choice of S_i

Improved analysis (Proof of Theorem 1.5)

Construct (possibly infeasible) dual solution:

Let i be the first iteration in which e is covered.

Set $y(e) := \dfrac{w(S_i)}{|S_i \setminus \bigcup_{j=1}^{i-1} S_j|}$. Note that $\sum_{e \in E} y(e) = \sum_{i=1}^{\ell} w(S_i)$.

Show that $\sum_{e \in S} y(e) \leq H_{|S|} \cdot w(S) \quad \forall S \in \mathcal{F}$ (see book).

Then $\frac{1}{H_g} \cdot y$ is a feasible dual solution of value $\frac{1}{H_g} \cdot \sum_{i=1}^{\ell} w(S_i)$. $\square$

("dual fitting")