Theorem 4.2 Algorithm 1 solves KNAPSACK in time $O(n \cdot \min(B, V))$.

Proof By induction: For every $X \subseteq [j]$ with $\sum_{i \in X} s_i \leq B$ there is $X' \in A(j)$ with $X' \succeq X$.
This is trivial for $j = 0$. Now let $j > 0$.

If $j \notin X$ then $X \subseteq [j-1]$ and there is $X' \in A(j-1)$ with $X' \succeq X$ by IH.
If $X' \notin A(j)$ then there is $X'' \in A(j)$ with $X'' \succeq X' \succeq X$.

If $j \in X$ then let $\bar{X} = X \setminus \{j\} \subseteq [j-1]$. There is $\bar{X}' \in A(j-1)$ with $\bar{X}' \succeq \bar{X}$ by IH.
Note that $\sum_{i \in \bar{X}'} s_i + s_j \leq \sum_{i \in \bar{X}} s_i + s_j = \sum_{i \in X} s_i \leq B$. Hence there is $\bar{X}'' \in A(j)$ with

$\bar{X}'' \succeq \bar{X}' \cup \{j\} \succeq \bar{X} \cup \{j\} = X$. [induction complete]

Consider optimal solution X^*. By the above there is $X \in A(n)$ with $s(X) \leq s(X^*)$
and $v(S) \geq v(S^*)$.

Running time: $s(X) \in \{0, \dots, B\}\ \forall X \in A(j) \xRightarrow[\text{(Lem 4.1)}]{\text{no dominated } X \text{ in } A(j)} |A(j)| \leq \min\{B, V\}$ $\square$

This running time is pseudo-polynomial.
encoding size: $\log_2(B)$, $\log_2(V)$

Lemma 4.1
- $X \preceq X' \wedge X' \preceq X'' \Rightarrow X \preceq X''$
- $s(X) = s(X') \Rightarrow X \preceq X'$ or $X' \succeq X$

Theorem 4.3 Algorithm 2 is a $(1-\varepsilon)$-approximation for KNAPSACK. It runs in time $O(\frac{n^3}{\varepsilon})$.

Proof Running time: $O(n V')$ where $V' := \sum_{i \in [n]} v_i' = \sum_{i \in [n]} \lfloor \frac{v_i}{M} \cdot \frac{n}{\varepsilon} \rfloor \leq \frac{n^2}{\varepsilon}$.

Approximation factor: $\sum_{i \in X} v_i \geq \mu \sum_{i \in X} v_i' \geq \mu \sum_{i \in O} v_i' \geq \mu \sum_{i \in O} \left(\frac{v_i}{\mu} - 1\right) = \sum_{i \in O} v_i - \mu |O|$

$$= \underbrace{\sum_{i \in O} v_i}_{= OPT} - \varepsilon \underset{\leq OPT}{M} \geq (1-\varepsilon) \cdot OPT \quad \square$$

Example — FPTAS vs PTAS

$O(\frac{n^3}{\varepsilon}) \to$ FPTAS $\qquad O(n^3 \cdot 2^{1/\varepsilon}) \to$ PTAS, but no FPTAS

Example for Algo 1

$s_1 = 2,\ v_1 = 5 \quad B = 4$
$s_2 = 2,\ v_2 = 2$
$s_3 = 3,\ v_3 = 5$

$A(0) = \{\emptyset\}$
$A(1) = \{\underset{0,0}{\emptyset}, \underset{2,5}{\{1\}}\}$
$A(2) = \{\underset{0,0}{\emptyset}, \underset{2,5}{\{1\}}, \underset{2,2}{\cancel{\{2\}}}, \underset{4,7}{\{1,2\}}\}$
$A(3) = \{\emptyset, \{1\}, \underline{\{1,2\}}, \underset{3,5}{\cancel{\{3\}}}\}$