A PTAS for $P||C_{max}$

job j long: $p_j \geq \frac{1}{km} \sum_{j \in [n]} p_j$

$i^* \in \underset{i \in [m]}{\operatorname{argmax}}\ load_\sigma(i)$, j^*: last job assigned to i^*

Proof of Thm. 5.1 (Algorithm $A(k)$)

If j^* is long then $ALG = OPT$.

If j^* is short: $ALG \underset{\substack{\text{all machines busy} \\ \text{at start of } j^*}}{\leq} \frac{1}{m} \sum_{j \in [n]} p_j + p_{j^*} \underset{j^* \text{ short}}{\leq} \left(1 + \frac{1}{k}\right) \cdot \frac{1}{m} \sum_{j \in [n]} p_j$

If we restrict $P||C$ to instances with a constant number of machines then running $A(\lfloor \frac{1}{\varepsilon} \rfloor)$ is a PTAS.

Arbitrary number of machines

Proof of Thm. 5.2 Can assume $T \geq \frac{1}{m} \sum_{j \in [n]} p_j$.

If j^* is long then $ALG \leq T$.

If j^* is short: $ALG \underset{\substack{\text{all machines busy} \\ \text{at start of } j^*}}{\leq} \frac{1}{m} \sum_{j \in [n]} p_j + p_{j^*} \underset{j^* \text{ short}}{\leq} \left(1 + \frac{1}{k}\right) T$

Proof of Lemma 5.3

1. If $B(k,T)$ returns "failed": Note that $p_j' \leq p_j \ \forall j$.

If we cannot schedule long jobs on m machines in time T for processing times p_j', we also cannot do it for p_j. $\Rightarrow T < OPT$

2. If $B(k,T)$ returns schedule σ:

(i) $p_j \leq p_j' + \frac{T}{k^2} \ \forall j$

(ii) $|\{j : \sigma(j) = i\}| \leq k \quad \forall i \in [m]$ $\left(\text{because } \underset{\text{all jobs are long}}{p_j'} \geq k \cdot \frac{T}{k^2} = \frac{T}{k}\right)$

$$\Rightarrow \sum_{j : \sigma(j) = i} p_j \overset{(i)}{\leq} \sum_{j : \sigma(j) = i} \left(p_j' + \frac{T}{k^2}\right) \overset{(ii)}{\leq} T + \frac{T}{k} \quad \square$$

Dynamic Program (Lemma 5.4) Can assume $\max p_j \leq T$.

There are at most k^2 different processing times, $p_j' \in \{\frac{T}{k^2}, 2\frac{T}{k^2}, \ldots, k^2\frac{T}{k^2}\}$.

job j is of type $i :\Longleftrightarrow p_j' = i \cdot \frac{T}{k^2}$ (no jobs of type $i<k$)

Configuration: $s \in \mathbb{Z}_+^{k^2}$, $\sum_{i=1}^{k^2} s_i \cdot i \cdot \frac{T}{k^2} \leq T$, $0 \leq s_i \leq k$

s_i ← number of jobs of type i on that machine

$C :=$ set of all configurations, $|C| \leq (k+1)^{k^2}$

Want to compute

$M(n_1, \ldots, n_{k^2}) :=$ min number of machines to schedule instance with n_i jobs of type i

for every $n_1 \in \{0, \ldots, n\}, \ldots, n_{k^2} \in \{0, \ldots, n\}$. ($n^{k^2}$ entries)

$$M(0, \ldots, 0) = 0 \quad M(n_1, \ldots, n_{k^2}) = 1 + \min_{s \in C} M(n_1 - s_1, \ldots, n_{k^2} - s_{k^2}).$$

$M(n_1, \ldots, n_{k^2})$ can be computed in time $O(|C|)$ if $M(n_1', \ldots, n_{k^2}')$ is known for all $n' \leq n$.