Lecture: Approximation Algorithms

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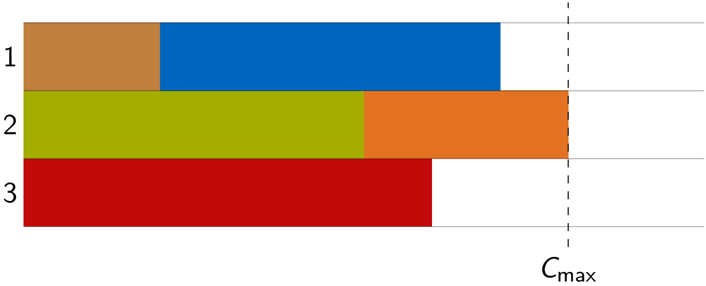
Dynamic Programming

Example II: Scheduling on Identical Parallel Machines

Scheduling on Parallel Machines

Input: m identical machines,
n jobs with processing times $p_1, \dots, p_n$

Task: assign each job $j \in [n]$ to a machine $\sigma(j) \in [m]$
minimizing $C_{\max} := \max_{i \in [m]} \sum_{j\,:\,\sigma(j)=i} p_j$



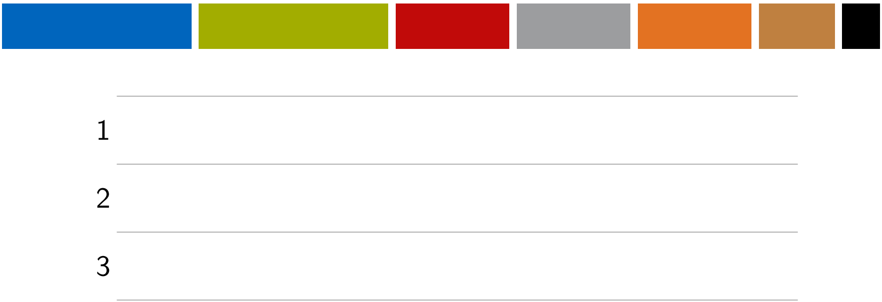
An approximation scheme

Algorithm $A(k)$

j' long: $p_{j'} \geq \frac{1}{km} \sum_{j \in [n]} p_j$

1. Compute optimal schedule for long jobs. (complete enumeration)
2. For each short job j

    Assign j to machine i with lowest load.



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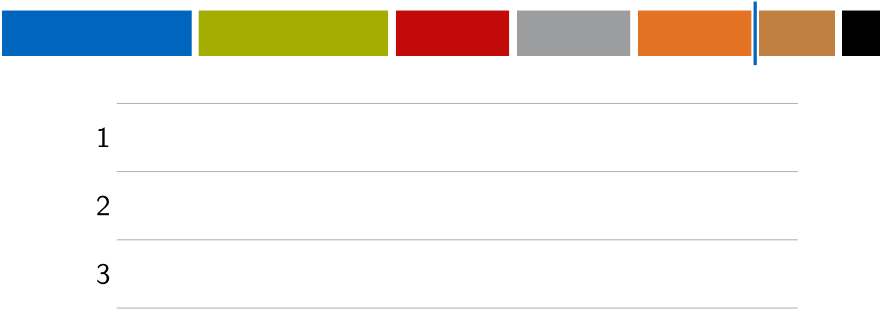

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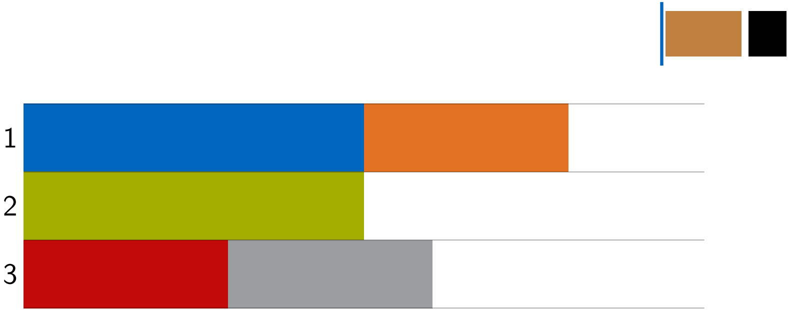

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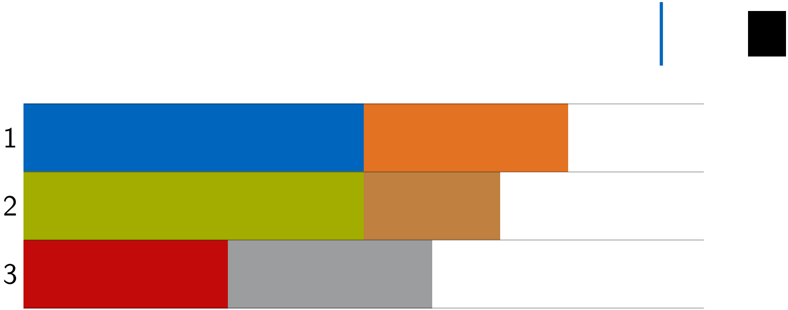

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Theorem 5.5

$A(k)$ is a $(1 + 1/k)$-approximation with running time $O(m^{km} + n)$.

An approximation scheme

Algorithm $A'(k, T)$

j long: $p_j \geq \frac{T}{k}$

1. Schedule long jobs within $(1 + 1/k)T$ (or find out that $T < \mathrm{OPT}$ and stop).
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If $A'(k, T)$ computes a schedule, then it has makespan $(1 + 1/k)T$.

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Subroutine for long jobs

Algorithm $B(k, T)$ only long jobs: $p_j \geq \frac{T}{k}$

1. $p'_j := \left\lfloor \frac{k^2 p_j}{T} \right\rfloor \cdot \frac{T}{k^2}$ for each j
2. Compute $m' :=$ min number of machines needed to schedule rounded long jobs within makespan T.
3. If $m' \leq m$ then return corresponding schedule.
4. Otherwise return "failed".

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Lemma 5.7

If $B(k, T)$ computes a schedule, then it has makespan $(1 + 1/k)T$.
If $B(k, T)$ returns "failed", then $\mathrm{OPT} > T$.

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Lemma 5.8

$B(k, T)$ runs in time $O(n^{k^2}(k+1)^{k^2})$.

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$$\text{dynamic program: } n^{k^2}(k+1)^{k^2}$$

Dynamic program: Pseudocode

Algorithm DP

1. $M(0, \ldots, 0) := 0,\ Q := \{(0, \ldots, 0)\}$
2. while $(Q \neq \emptyset)$

    Let $q \in Q$ such that $M(q)$ is minimum.
    for each $s \in \mathcal{C}$ with $q_i + s_i \leq n$ for all i
        if $M(q) + 1 < M(q + s)$ then
            $M(q + s) := M(q) + 1$
            $Q := Q \cup \{q + s\}$
        end if
    end for
    $Q := Q \setminus \{q\}$

3. Return $M(n_1, \ldots, n_{k^2})$.

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$A'(k, T)$ is a $(1 + 1/k)$-relaxed decision procedure.

Can be turned into a $(1 + 1/k)$-approximation algorithm. (Exercise)

PTAS: choose $k := \lceil 1/\varepsilon \rceil$

Why no FPTAS?

Theorem 5.10

There is a polynomial function q such that $P||C_{\max}$ is NP-hard even when restricted to instances with $p_j \leq q(n)$ for all j.

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Run FPTAS with $\varepsilon := 1/(Q+1)$. $\mathrm{OPT} \leq Q$

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- poly-time in input size, because $1/\varepsilon \leq nq(n) + 1$
- $\mathrm{ALG} \leq (1+\varepsilon)\,\mathrm{OPT} < \mathrm{OPT} + \mathrm{OPT}/Q \leq \mathrm{OPT} + 1$
  by integrality: $\mathrm{ALG} \leq \mathrm{OPT}$. $\square$