Prize-collecting Steiner Tree

Solving the LP: Given (x,y) compute a min v-r-cut S_v w.r.t. to capacities x for every $v\in V$. If $\sum_{e\in\delta(S_v)} x(e) < y(v)$ for some v, we have found a violated inequality. Otherwise, (x,y) is a feasible solution. We can thus solve the separation problem for the LP and optimize (see Section 4.3 in book).

Analysis of Algorithm D

Proof of Claim 1: Note that $\frac{1}{\alpha}x^*$ is a feasible solution to

$$
\begin{aligned}
Z^*(U) := \min &\sum_{e\in E} d(e)x(e) \\
\text{s.t.} &\sum_{e\in\delta(S)} x(e) \geq \cancel{\tfrac{1}{\alpha}y^*(v)}\ 1 \quad \forall S\subseteq V,\ v\in S\cap U \\
& x(e)\geq 0 \quad \forall e\in E
\end{aligned}
$$

(LP relaxation for STEINER TREE)

T is an MST of $G[U\cup\{r\}]$. $\Rightarrow$ $d(T)\leq 2Z^*(U)$ (Problem Set 4)

$$\leq \frac{2}{\alpha}\sum_{e\in E} d(e)x^*(e) \quad \square$$

Proof of Claim 2: $(1-\alpha)\cdot\sum_{v\in V\setminus U}\pi(v) \leq \sum_{v\in V\setminus U}(1-y^*(e))\pi(v) \leq \sum_{v\in V}(1-y^*(v))\pi(v)$ $\square$

$y^*(v)<\alpha\ \forall v\in V\setminus U$

Claim 1 & 2: $d(T)+\pi(V\setminus U)\leq \max\left\{\frac{2}{\alpha},\frac{1}{1-\alpha}\right\}\cdot Z^* \leq 3\cdot OPT$

$\alpha = \frac{2}{3}$

Analysis of Algorithm R

Claim 1: $\mathbb{E}[d(T)] \leq \frac{-2\ln\gamma}{1-\gamma}\cdot\sum_{e\in E} d(e)x^*(e)$

Proof: $\mathbb{E}[d(T)] \leq \mathbb{E}\left[\frac{2}{\alpha}\cdot\sum_{e\in E} d(e)x^*(e)\right] = \mathbb{E}\left[\frac{2}{\alpha}\right]\sum_{e\in E} d(e)x^*(e)$

$$\mathbb{E}\left[\frac{2}{\alpha}\right] = \int_{t=\gamma}^{1}\frac{2}{t}\cdot\frac{1}{1-\gamma}\,dt = \left[\ln t\right]_{t=\gamma}^{1}\cdot\frac{2}{1-\gamma} = \frac{-2\ln\gamma}{1-\gamma} \quad \square$$

Claim 2: $\mathbb{E}[\pi(V\setminus U)] \leq \frac{1}{1-\gamma}\cdot\sum(1-y^*(v))\pi(v)$

Proof: $\mathbb{E}\left[\sum_{v\in V\setminus U}\pi(v)\right] = \sum_{v\in V}(1-\Pr[v\in U])\cdot\pi(v) = \sum_{v\in V}\frac{1-y^*(v)}{1-\gamma}\pi(v)$ $\square$

Claim 1 & 2 $\Rightarrow \mathbb{E}[d(T)+\pi(V\setminus\mathcal{U})] \leq \max\left\{\frac{-2\ln\gamma}{1-\gamma}, \frac{1}{1-\gamma}\right\}\cdot Z^*$.

$$= \frac{1}{1-\exp(-\frac{1}{2})}\cdot Z^* \quad \square$$

Uncapacitated Facility Location

$$\begin{aligned}
\min \ & \sum_{i\in F}\sum_{j\in C} d_{ij}x_{ij} + \sum_{i\in F} f_i y_i \\
\text{s.t. } & \sum_{i\in F} x_{ij} = 1 \quad \forall j\in C \\
& y_i - x_{ij} \geq 0 \quad \forall i\in F, j\in C \\
& x_{ij} \geq 0 \quad \forall i\in F, j\in C \\
& y_i \geq 0 \quad \forall i\in F
\end{aligned}$$

$$\begin{aligned}
\max \ & \sum_{j\in C} v_j \\
\text{s.t. } & v_j - w_{ij} \leq d_{ij} \quad \forall i\in F, j\in C \\
& \sum_{j\in C} w_{ij} \leq f_i \quad \forall i\in F \\
& w_{ij} \geq 0 \quad \forall i\in F, \forall j\in C
\end{aligned}$$

opt solutions (x^*, y^*) and (v^*, w^*)

Lemma 6.5 If $i\in N_j$ then $d_{ij} \leq v_j^*$.

Proof: $i\in N_j \Rightarrow x_{ij}^* > 0 \underset{\text{compl. slackness}}{\Rightarrow} v_j^* - w_{ij}^* = d_{ij} \underset{w_{ij}\geq 0}{\Rightarrow} v_j^* \geq d_{ij}$ $\square$

Lemma 6.6 If $i'\in N_j$ with $f_{i'} = \min_{i\in N_j} f_i$ then $f_{i'} \leq \sum_{i\in N_j} f_i y_i^*$.

Proof: $\sum_{i\in N_j} f_i y_i^* \underset{x_{ij}^*\leq y_i^*}{\geq} \sum_{i\in N_j} f_i x_{ij}^* \geq \sum_{i\in N_j} f_{i'} x_{ij}^* \underset{\sum_{i\in F} x_{ij}=1}{\geq} f_{i'}$ $\square$

Analysis of Algorithm D

Let $j_1,\dots,j_n$ be the clients selected by the algorithm and $\underbrace{i_1,\dots,i_n}_{S}$ be the corresponding facilities. Then:

$$\sum_{i\in S} f_i = \sum_{k=1}^{n} f_{i_k} \underset{\text{Lem. 6.6}}{\leq} \sum_{k=1}^{n}\sum_{i\in N_{j_k}} f_i y_i^* \underset{N_{j_k}\cap N_{j_{k'}}=\emptyset}{\leq} \sum_{i\in F} f_i y_i^*$$

Let $j\in C$ and let $k\in[n]$ be such $j\in N_{j_k}^2$. Then $v_j^* \geq v_{j_k}^*$.

Because $j\in N_{j_k}^2$, there is $i\in N_j\cap N_{j_k}$. Thus $d_{i_k j} \leq \underbrace{d_{i_k j_k}}_{\leq v_{j_k}^*} + \underbrace{d_{i j_k}}_{\leq v_{j_k}^*} + \underbrace{d_{ij}}_{\leq v_j^*} \leq 3v_j^*$ (Lem 6.5)

Thus $\sum_{i\in S} f_i + \sum_{j\in C} \min_{i\in S} d_{ij} \leq \sum_{i\in F} f_i y_i^* + \sum_{j\in C} 3 v_j^* \leq 4 Z^*$. $\square$

Analysis of Algorithm D* & R

$$\mathbb{E}\Big[\sum_{i\in S} f_i\Big] = \sum_{k=1}^{n} \mathbb{E}[f_{i_k}] = \sum_{k=1}^{n} \sum_{i\in N_{j_k}} f_i x^*_{ij_k} \leq \sum_{k=1}^{n} \sum_{i\in N_{j_k}} f_i y_i^* \leq \sum_{i\in F} f_i y_i^*$$

$$\mathbb{E}[d_{i_k j_k}] = \sum_{i\in N_{j_k}} d_{ij_k} x^*_{ij_k} = \Delta_{j_k}$$

Let $j\in C$ and $k\in[n]$ such that $j\in N^2_{j_k}$. Again, for some $i\in N_j \cap N_{j_k}$:

$$\mathbb{E}[d_{i_k j}] \leq \underbrace{\mathbb{E}[d_{i_k j_k}]}_{=\Delta_{j_k}} + \underbrace{d_{ij_k}}_{\leq v^*_{j_k}} + \underbrace{d_{ij}}_{\leq v^*_j} \leq v_j^* + v^*_{j_k} + \Delta_{j_k} \leq 2v_j^* + \Delta_j$$

(Lem. 6.5) $\quad$ choice of j_k

Hence: $\mathbb{E}\Big[\sum_{i\in S} f_i + \sum_{j\in C} \min_{i\in S} d_{ij}\Big] \leq \sum_{i\in F} f_i y_i^* + \sum_{i\in C} 2v_j + \Delta_j \leq 3Z^*$ $\square$