Primal-dual method for SHORTEST PATH

$$\text{(D)} \quad \max \sum_{S \in \mathcal{S}'} y_S$$
$$\text{s.t.} \sum_{S \in \mathcal{S}_e} y_S \le w_e \quad \forall\, e \in E$$
$$y_S \ge 0 \quad \forall\, S \in \mathcal{S}'$$

$$\mathcal{S}_e := \{S \in \mathcal{S}' : e \in \delta(S)\}$$

<u>Lemma</u> Throughout the algorithm, T is a tree containing s.

<u>Proof</u>: True initially with $T = \emptyset$.
When adding edge e, then $e \in \delta(C)$ for the current connected component C spanned by T. Thus e does not close a cycle and $T \cup \{e\}$ is connected. $\square$

<u>Proof of Theorem 9.1:</u>

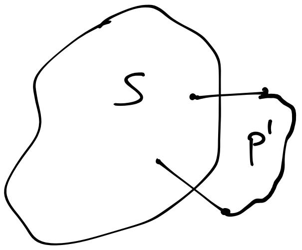
$$\sum_{e \in P} w_e \underset{\text{by choice of } e}{=} \sum_{e \in P} \sum_{S \in \mathcal{S}_e} y_S = \sum_{S \in \mathcal{S}'} |P \cap \delta(S)|\, y_S \underset{\text{Claim}}{=} \sum_{S \in \mathcal{S}'} y_S \le Z^* \le \text{OPT} \quad \square$$

$\uparrow$ y feasible for (D)

<u>Claim</u>: $|P \cap \delta(S)| = 1 \quad \forall\, S \in \mathcal{S}'$ with $y_S > 0$.

<u>Proof</u> Assume $|P \cap S| \ge 2$ for some $S \in \mathcal{S}$ with $y_S > 0$.

$y_S > 0 \implies S = C$ in one iteration of the algorithm.



By Lemma, T was a tree spanning S at the beginning of that iteration, i.e., $T[S] := \{e \in T : e \subseteq S\}$ spans S. $^{(1)}$

$|P \cap \delta(S)| \ge 2 \implies \exists$ subpath $P' \subseteq P$ starting with only start and end vertex in S, but all other vertices in $V \setminus S$. $^{(2)}$

$(1) \,\&\, (2) \implies P' \cup T[S] \subseteq T$ contains a cycle. $\lightning$

Primal-dual method for STEINER FOREST

$$(D)\quad \max \sum_{S\in\mathcal{S}} y_S \qquad\qquad \mathcal{S}_e := \{S\in\mathcal{S} : e\in\delta(S)\}$$

$$\text{s.t.} \sum_{S\in\mathcal{S}_e} y_S \le w_e \quad \forall e\in E$$

$$y_S \ge 0 \quad \forall S\in\mathcal{S}$$

Lemma Let $\bar F$ be the forest returned by the algorithm.
Throughout the algorithm: $\sum_{C\in\mathcal{C}} |\delta(C)\cap\bar F| \le 2|\mathcal{C}|$.

Proof of Theorem 9.1

$$\sum_{e\in\bar F} w_e = \sum_{e\in\bar F}\sum_{S\in\mathcal{S}_e} y_S = \sum_{S\in\mathcal{S}} |\bar F\cap\delta(S)|\, y_S \underset{\text{Claim}}{\le} 2\sum_{S\in\mathcal{S}} y_S \le 2z^* \quad \square$$

Claim Throughout the algorithm: $\sum_{S\in\mathcal{S}} |\bar F\cap\delta(S)|\, y_S \le 2\sum_{S\in\mathcal{S}} y_S$

Proof: True initially with $y=0$.

When increasing all y_C for $C\in\mathcal{C}$ by ε:

- LHS increases by $\sum_{C\in\mathcal{C}} |\bar F\cap\delta(C)|\cdot\varepsilon$
- RHS increases by $2|\mathcal{C}|\cdot\varepsilon$

By Lemma, RHS increases at least as much as RHS.

$\Rightarrow$ Invariant is maintained. $\boxtimes$

not discussed in class

Proof of Lemma: Let F be the current forest in the algorithm.

Consider graph $H=(\bar V, E')$ arising from contracting all edges of F.

Identify every $C\in\mathcal{C}$ with a vertex $v_C\in\bar V$. $\quad R:=\{v_C : C\in\mathcal{C}\}$

Identify every $f\in\bar F\setminus F$ with an edge $e_f\in E'$. $\quad B:=\bar V\setminus R$

$F^* := \{e_f : f\in\bar F\setminus F\}$

F^* is a forest in H and $\sum_{C\in\mathcal{C}} |\bar F\cap C| = \sum_{v\in R} |\delta_{F^*}(v)|$.

Claim: Let $v \in \bar{V}$ with $|\delta_{F^*}(v)| = 1$. Then $v \in R$.

Proof: Let $f \in \bar{F}$ such that $\{e_f\} = \delta_{F^*}(v)$.

By Step 2 of Alg.: $\exists i$: f is contained in s_i-t_i-path in $\bar{F}$

$\Rightarrow$ v corresponds to conn. comp. C of F with $|\{s_i, t_i\} \cap C| = 1$.

$\Rightarrow$ $C \in \mathcal{C}$ and $v \in R$ $\boxtimes$

Thus, for every $v \in B$ either $\delta_{F^*}(v) = 0$ or $\delta_{F^*}(v) \geq 2$.

Let $\bar{B} := \{v \in B : \delta_{F^*}(v) \geq 2\}$. Note that $|F^*| \leq |R| + |\bar{B}| - 1$.

$$\sum_{v \in R} \delta_{F^*}(v) = \underbrace{\sum_{v \in \bar{V}} \delta_{F^*}(v)}_{= 2 \cdot |F^*|} - \underbrace{\sum_{v \in B} \delta_{F^*}(v)}_{\substack{\geq 2|\bar{B}| \\ \text{by claim}}} \leq 2|F^*| - 2|\bar{B}| \leq 2|R|$$

$$\Rightarrow \sum_{C \in \mathcal{C}} |\bar{F} \cap \delta(C)| = \sum_{v \in R} \delta_{F^*}(v) \leq 2|R| = 2|\mathcal{C}| \quad \square$$