Uncapacitated Facility Location

Dual LP: $\max \sum_{j\in C} v_j$

$$\text{s.t. } v_j - w_{ij} \le d_{ij} \quad \forall\, i\in F, j\in C$$

$$\sum_{j\in C} w_{ij} \le f_i \quad \forall\, i\in F$$

$$w_{ij} \ge 0 \quad \forall\, i\in F, j\in C$$

<u>Lemma 10.1</u> (v,w) computed by primal-dual algorithm is a feasible dual solution.

<u>Lemma 10.2</u> $\sum_{i\in\bar S} f_i + \frac{1}{3}\sum_{j\in C} d(\bar S, j) \le \sum_{j\in C} v_j$

<u>Proof</u>: Let $\bar C := \bigcup_{i\in\bar S} N_i$.

<u>Claim 1</u>: $\sum_{i\in\bar S} f_i + \sum_{j\in\bar C} d(\bar S, j) \le \sum_{j\in\bar C} v_j$

Proof: $\sum_{i\in\bar S} f_i = \sum_{i\in\bar S}\sum_{j\in N_i} w_{ij} = \sum_{i\in\bar S}\sum_{j\in N_i}(v_j - d_{ij})$

$$\le \sum_{i\in\bar S}\sum_{j\in N_i}(v_j - d(\bar S, j)) \underset{N_i\cap N_{i'}=\emptyset}{=} \sum_{j\in\bar C} v_j - d(\bar S, j) \quad \boxtimes$$

<u>Claim 2</u>: $d(\bar S, j) \le 3 v_j$ for every $j\in C\setminus\bar C$

<u>Proof</u>: Let $i\in S$ be the facility that is responsible for j's removal from C'. Note that $v_j \ge d_{ij}$. If $i\in\bar S$ then $d(\bar S, j)\le d_{ij}\le v_j$. If $i\notin\bar S$ then there is $i'\in\bar S$ with $N_i\cap N_{i'}\neq\emptyset$. Let $j'\in N_i\cap N_{i'}$. By triangle inequality:

$$d_{i'j} \le \underbrace{d_{i'j'}}_{<v_{j'}} + \underbrace{d_{ij'}}_{<v_{j'}} + d_{ij} \le 2v_{j'} + v_j \quad \text{because } j'\in N_i\cap N_{i'}$$

Because $j'\in N_i$, client j' was removed from C' at latest when i was added to S. Furthermore, j was not removed from C' before i was added to S.

$\Rightarrow v_j \ge v_{j'}$ $\boxtimes$

Claim 1 & 2 $\Rightarrow$ Lemma $\square$

<u>Theorem 10.3</u> Primal-dual is a 3-approximation for UFL.

The k-MEDIAN Problem

Lemma 10.4 $g(\lambda) \leq Z^* \quad \forall \lambda \geq 0$

Proof: Let (x^*, y^*) be the optimal solution to [P].

Then (x^*, y^*) is feasible solution to $[P(\lambda)]$ and $\lambda\left(\sum_{i \in F} y_i - k\right) \leq 0$. $\square$

Observation Let (v, w) be a feasible solution to $[D(\lambda)]$.

Then $\sum_{j \in C} v_j \leq g(\lambda) + k\lambda \leq OPT + k\lambda$.

Lemma 10.5 Primal-dual computes $S \subseteq F$ with $\sum_{j \in C} d(S, j) \leq 3(OPT + (k - |S|)\lambda)$.

Proof: Analysis of Primal-Dual $\Rightarrow 3 \cdot \underbrace{\sum_{i \in S} f_i}_{= |S|\lambda} + \sum_{j \in C} d(S, j) \leq 3 \sum_{j \in C} v_j$

$$\Rightarrow \sum d(S, j) \leq 3\left(OPT + k\lambda - |S|\lambda\right) \quad \square$$

Define $f(S) := \sum_{j \in C} d(S, j)$. Let $\alpha := \frac{k - |S_2|}{|S_1| - |S_2|}$.

Lemma 10.6 $\alpha |S_1| + (1 - \alpha)|S_2| = k$ and $\alpha f(S_1) + (1 - \alpha) f(S_2) \leq (3 + \varepsilon) \cdot OPT$

Proof:
$$\frac{(k - |S_2|)|S_1|}{|S_1| - |S_2|} + \frac{(|S_1| - |S_2| - k + |S_2|) \cdot |S_2|}{|S_1| - |S_2|} = \frac{k(|S_1| - |S_2|)}{|S_1| - |S_2|}$$

$$f(S_1) \leq 3(OPT + (k - |S_1|)\lambda_1) = 3\left(OPT + k\lambda_1 - |S_1|\left(\underbrace{\lambda_1 - \lambda_2}_{\leq \frac{\varepsilon \cdot OPT}{3|F|}} + \lambda_2\right)\right)$$
$$\leq 3(OPT + k\lambda_1 - |S_1|\lambda_2) + \varepsilon \cdot OPT$$

$\alpha f(S_1) + (1 - \alpha) f(S_2)$

$$\leq 3\left(\alpha(OPT + k\lambda_1 - |S_1|\lambda_2) + (1 - \alpha)(OPT + k\lambda_2 - |S_2|\lambda_2)\right) + \alpha\varepsilon\, OPT$$

$$\leq 3\left(OPT + k\underbrace{(\alpha\lambda_1 + (1-\alpha)\lambda_2)}_{\leq \lambda_2} - \underbrace{(\alpha|S_1| + (1-\alpha)|S_2|)}_{= k}\lambda_2\right) + \alpha\varepsilon\, OPT$$

$$\leq (3 + \varepsilon)\, OPT. \quad \square$$

Proof of Theorem 10.7:

Case $\alpha \leq \frac{1}{2}$: $\quad |S_2| \leq k$ and $\frac{1}{2} f(S_2) \leq (1-\alpha) f(S_2) \leq (3+\varepsilon)\mathrm{OPT}$

Case $\alpha > \frac{1}{2}$: "nearest facility": For $i, i' \in F$ define $d_{ii'} := \min\{d_{ij} + d_{i'j} : j \in C\}$.

Let $j \in C$. Let $i_1 \in S_1$ with $d_{i_1 j} = d(S_1, j)$ and $i_2 \in S_2$ with $d_{i_2 j} = d(S_2, j)$.

Either $i_1 \in \bar{S}$ or $i_1 \in X$ with probability $\frac{k - |\bar{S}|}{|S_1| - |\bar{S}|} \geq \frac{k - |S_2|}{|S_1| - |S_2|} = \alpha$.

Furthermore $d(\bar{S}, j) \leq d_{\phi(i_2) j} \leq d_{i_2 j} + d_{i_2 \phi(i_2)} \leq d_{i_2 j} + d_{i_1 i_2} \leq d_{i_2 j} + d_{i_1 j} + d_{i_2 j}$.

$$\Rightarrow \mathbb{E}[d(\bar{S} \cup X, j)] \leq \alpha d_{i_1 j} + (1-\alpha)(d_{i_1 j} + 2 d_{i_2 j}) \underset{\alpha > \frac{1}{2}}{\leq} 2(\alpha d(S_1, j) + (1-\alpha) d(S_2, j))$$

$$\leq (6 + 2\varepsilon)\mathrm{OPT} \quad \square$$

Remark: Construction of X can be derandomized.