The k-MEDIAN Problem

Lemma 10.5 Primal-dual computes $S \subseteq F$ with $\sum_{j \in C} d(S,j) \leq 3(OPT + (k-|S|)\lambda)$.

Define $f(S) := \sum_{j \in C} d(S,j)$. Let $\alpha := \frac{k-|S_2|}{|S_1|-|S_2|}$.

Lemma 10.6 $\alpha |S_1| + (1-\alpha)|S_2| = k$ and $\alpha f(S_1) + (1-\alpha) f(S_2) \leq (3+\varepsilon) \cdot OPT$

Proof:
$$\frac{(k-|S_2|)|S_1|}{|S_1|-|S_2|} + \frac{(|S_1|-|S_2|-k+|S_2|)\cdot|S_2|}{|S_1|-|S_2|} = \frac{k(|S_1|-|S_2|)}{|S_1|-|S_2|}$$

$$f(S_1) \leq 3(OPT + (k-|S_1|)\lambda_1) = 3(OPT + k\lambda_1 - |S_1|(\underbrace{\lambda_1 - \lambda_2}_{\leq \frac{\varepsilon \cdot OPT}{3|F|}} + \lambda_2))$$
$$\leq 3(OPT + k\lambda_1 - |S_1|\lambda_2) + \varepsilon \cdot OPT$$

$$\begin{aligned}
\alpha f(S_1) + (1-\alpha) f(S_2) &\leq \underbrace{\alpha\varepsilon}_{\leq \varepsilon} OPT + 3\alpha(OPT + k\lambda_1 - |S_1|\lambda_2) \\
&\quad + 3(1-\alpha)(OPT + k\lambda_2 - |S_2|\lambda_2)) \\
&\leq (3+\varepsilon) OPT + 3(k \cdot \underbrace{(\alpha\lambda_1 + (1-\alpha)\lambda_2)}_{\leq \lambda_2} \\
&\quad - \lambda_2 \underbrace{(\alpha|S_1| + (1-\alpha)|S_2|)}_{=k})) \\
&\leq (3+\varepsilon) OPT. \quad \square
\end{aligned}$$

Theorem 10.7 The algorithm is a $(6+\varepsilon)$-approximation for k-MEDIAN.

Proof: Case $\alpha \leq \frac{1}{2}$: $|S_2| \leq k$ and $\frac{1}{2} f(S_2) \leq (1-\alpha) f(S_2) \leq (3+\varepsilon) OPT$

Case $\alpha > \frac{1}{2}$: "nearest facility": For $i, i' \in F$ define $d_{ii'} := \min\{d_{ij} + d_{i'j} : j \in C\}$.

Let $j \in C$. Let $i_1 \in S_1$ with $d_{i_1 j} = d(S_1, j)$ and $i_2 \in S_2$ with $d_{i_2 j} = d(S_2, j)$.

Either $i_1 \in \bar{S}$ or $i_1 \in X$ with probability $\frac{k-|\bar{S}|}{|S_1|-|\bar{S}|} \geq \frac{k-|S_2|}{|S_1|-|S_2|} = \alpha$.

Furthermore $d(\bar{S}, j) \leq d_{\phi(i_2)j} \leq d_{i_2 j} + d_{i_2 \phi(i_2)} \leq d_{i_2 j} + d_{i_1 i_2} \leq d_{i_2 j} + d_{i_1 j} + d_{i_2 j}$.

$\Rightarrow \mathbb{E}[d(\bar{S} \cup X, j)] \leq \alpha d_{i_1 j} + (1-\alpha)(d_{i_1 j} + 2 d_{i_2 j}) \underset{\Leftarrow \alpha > \frac{1}{2}}{\leq} 2(\alpha d(S_1, j) + (1-\alpha) d(S_2, j))$

$\Rightarrow \mathbb{E}[\sum_{j \in C} d(\bar{S} \cup X, j)] \leq (6+2\varepsilon) OPT$ $\square$

Remark: Construction of X can be derandomized.

SURVIVABLE NETWORK DESIGN

Lemma 11.1 LP(F) can be solved in polynomial time.

Proof: Separation routine, given $x \in \mathbb{R}_+^{E \setminus F}$:

Define capacities $u_e := \begin{cases} 1 & \text{if } e \in F \\ x_e & \text{if } e \in E \setminus F \end{cases}$.

For each $\{v, w\} \subseteq V$, compute min v-w-cut $\delta(S)$ w.r.t. u_e.

If $\sum_{e \in \delta(S)} u_e < r_{vw}$, then $\sum_{e \in \delta(S) \setminus F} x_e < f(S) - |\delta(S) \cap F|$. That is, we have found a separating inequality. Otherwise, x is feasible. $\square$

Lemma 11.2 The algorithm maintains the following invariant for the optimal solution x to LP(F):
$$\frac{1}{2} \sum_{e \in F} d_e + \sum_{e \in E \setminus F} d_e x_e \le OPT$$

Proof: By induction on the algorithm. True initially as $LP(\emptyset)$ is LP relaxation for SND. Now assume claim is true for F' and opt solution x' to $LP(F')$. Let $F^+ := \{ e \in E \setminus F' : x_e \geq \frac{1}{2} \}$ and $F := F' \cup F^+$.

Let x be an optimal solution to LP(F). Observe that $x'|_{E \setminus F}$ is a feasible solution to LP(F), because
$$f(S) - |\delta(S) \cap F'| \le \sum_{e \in \delta(S) \setminus F'} x'_e \le \sum_{e \in \delta(S) \setminus F} x'_e + |\delta(S) \cap F^+| \quad \forall S \subseteq V.$$

Thus, $\sum_{e \in E \setminus F} d_e x_e \le \sum_{e \in E \setminus F} d_e x'_e$ and it follows that
$$\frac{1}{2} \sum_{e \in F} d_e + \sum_{e \in E \setminus F} d_e x_e \le \frac{1}{2} \sum_{e \in F'} d_e + \frac{1}{2} \sum_{e \in F^+} d_e + \sum_{e \in E \setminus F} d_e x'_e$$
$$\le \frac{1}{2} \sum_{e \in F'} d_e + \sum_{e \in E \setminus F'} d_e x'_e \le OPT. \quad \square$$

It remains to show that the algorithm terminates.

$\longrightarrow$ next lecture