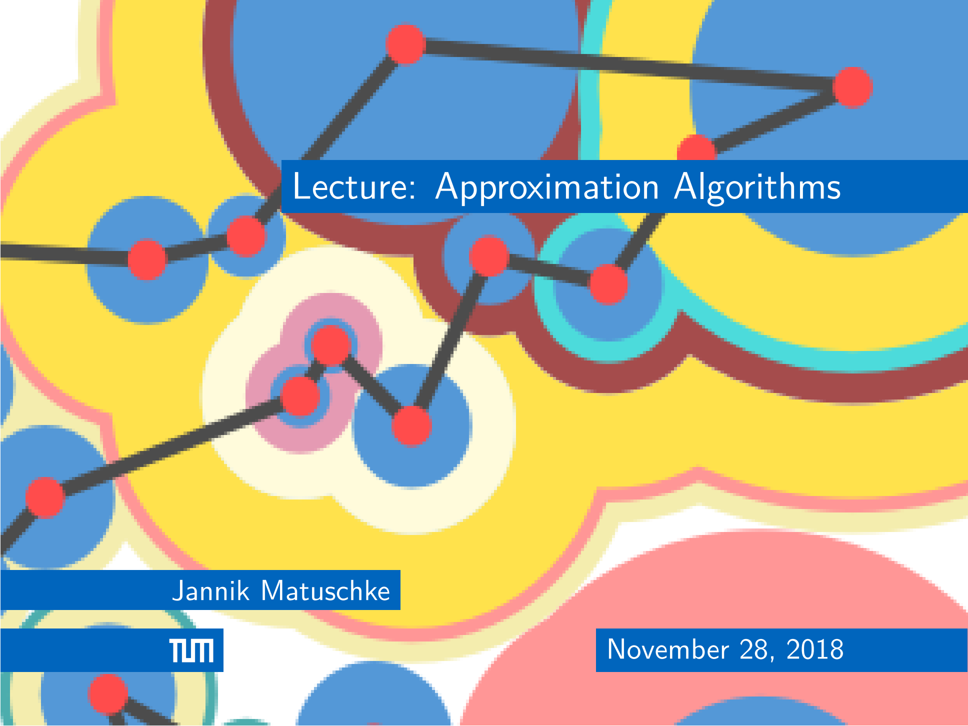

Lecture: Approximation Algorithms

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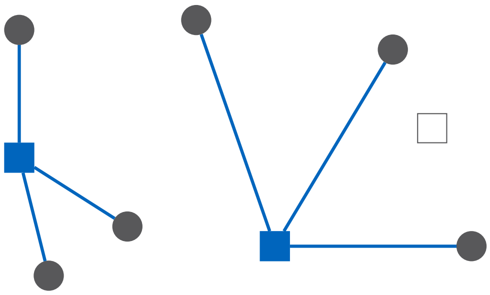
The primal-dual method and Lagrangean relaxation for k-Median (continued)

k-Median

Input: facilities F, clients C, $k \in \mathbb{N}$
metric distances d_{ij} for $i \in F$ and $j \in C$

Task: find $S \subseteq F$, $|S| \leq k$ minimizing $\sum_{j \in C} d(S, j)$



Lagrangean relaxation

$$
\begin{aligned}
\min \quad & \sum_{i \in F} \sum_{j \in C} d_{ij} x_{ij} \\
\text{s.t.} \quad & \sum_{i \in F} x_{ij} = 1 & \forall\, j \in C \\
& y_i - x_{ij} \geq 0 & \forall\, i \in F, j \in C \\
& \sum_{i \in F} y_i \leq k \\
& x_{ij} \geq 0 & \forall\, i \in F, j \in C \\
& y_i \geq 0 & \forall\, i \in F
\end{aligned}
$$

Lagrangean relaxation

$$\begin{aligned}
\min \quad & \sum_{i\in F}\sum_{j\in C} d_{ij}x_{ij} + \lambda\left(\sum_{i\in F} y_i \; - \; k\right) && \\
\text{s.t.} \quad & \sum_{i\in F} x_{ij} = 1 && \forall\, j \in C \\
& y_i - x_{ij} \geq 0 && \forall\, i \in F, j \in C \\
& \cancel{\sum_{i\in F} y_i \leq k} && \\
& x_{ij} \geq 0 && \forall\, i \in F, j \in C \\
& y_i \geq 0 && \forall\, i \in F
\end{aligned}$$

Lagrangean relaxation

$$\begin{aligned}
\min \quad & \sum_{i\in F}\sum_{j\in C} d_{ij}x_{ij} + \lambda\left(\sum_{i\in F} y_i \; - \; k\right) & \\
\text{s.t.} \quad & \sum_{i\in F} x_{ij} \; = 1 & \forall\, j \in C \\
& y_i - x_{ij} \; \geq \; 0 & \forall\, i \in F, j \in C \\
& \cancel{\sum_{i\in F} y_i \; \leq \; k} & \\
& x_{ij} \; \geq \; 0 & \forall\, i \in F, j \in C \\
& y_i \; \geq \; 0 & \forall\, i \in F
\end{aligned}$$

Idea: Choose some $\lambda \geq 0$. Run primal-dual algorithm for UFL instance with facility costs $f_i = \lambda$ for all $i \in F$.

Combining two solutions

Bisection search

We can find in polynomial time $\lambda_1, \lambda_2 \geq 0$ and corresponding $S_1, S_2 \subseteq F$ computed by the primal-dual algorithm such that

- $0 \leq \lambda_2 - \lambda_1 \leq \frac{\varepsilon \, \mathrm{OPT}}{3|F|}$ and
- $|S_1| \geq k \geq |S_2|$.

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Algorithm

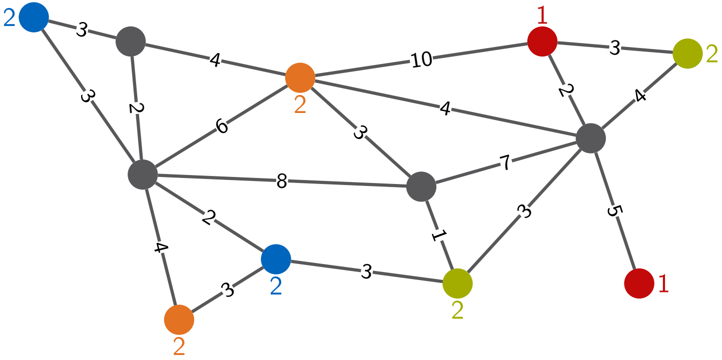
1. Find λ_1, S_1 and λ_2, S_2 as above. Define $\alpha := \dfrac{k - |S_2|}{|S_2| - |S_1|}$.
2. If $\alpha \leq 1/2$ then return S_2.
3. If $\alpha > 1/2$ then
    - For $i \in S_2$ let $\phi(i)$ be the nearest facility to i in S_1.
    - Let $\bar{S} := \{\phi(i) : i \in S_2\}$.
    - Select $X \subseteq S_1 \setminus \bar{S}$ with $|X| = k - |\bar{S}|$ uniformly at random.
    - Return $|\bar{S} \cup X|$.

Iterated Rounding for Survivable Network Design

Survivable Network Design

Input: graph $G = (V, E)$, weights $w : E \to \mathbb{R}_+$,
connectivity requirements r_{vw} for $\{v, w\} \subseteq V$

Task: find $F \subseteq E$ containing r_{vw} edge-disjoint v-w-paths
for every $\{v, w\} \subseteq V$, minimizing $\sum_{e \in F} w_e$



Survivable Network Design

Input: graph $G = (V, E)$, weights $w : E \to \mathbb{R}_+$, connectivity requirements r_{vw} for $\{v, w\} \subseteq V$

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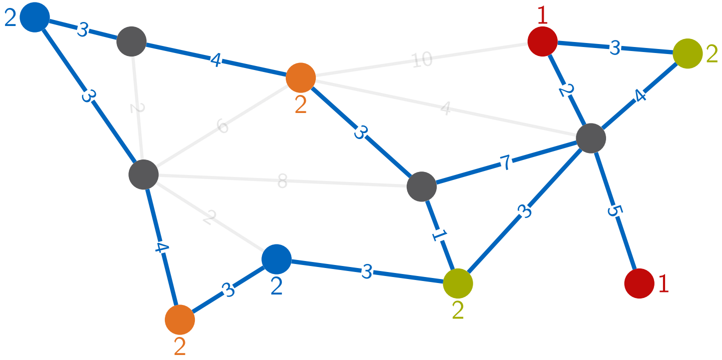
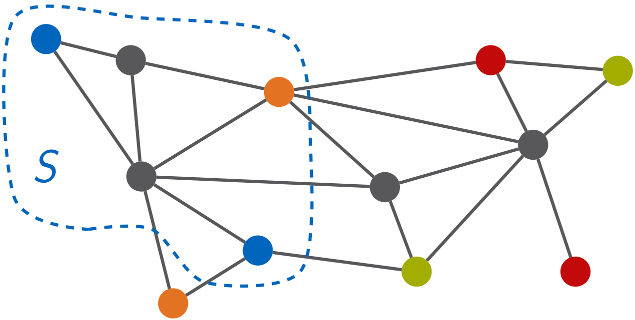


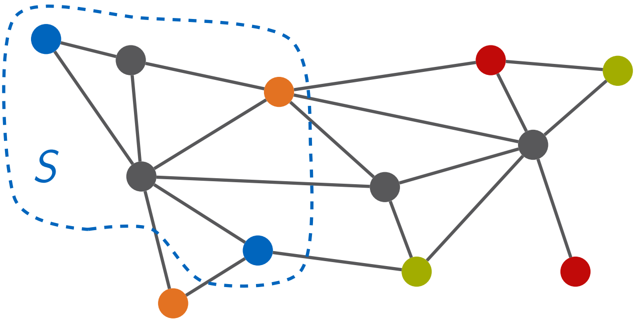
LP relaxation



$$f(S) := \max\{r_{vw} \ : \ v \in S,\, w \in V \setminus S\}$$

$$\begin{aligned} \min \ \ & \sum_{e \in E} w_e x_e \\ \text{s.t.} \ \ & \sum_{e \in \delta(S)} x_e \ \geq \ f(S) \quad \forall\, S \subseteq V \\ & \qquad\quad x_e \ \in \{0,1\} \quad\ \ \forall\, e \in E \end{aligned}$$

LP relaxation



$$f(S) := \max\{r_{vw} \ : \ v \in S, \, w \in V \setminus S\}$$

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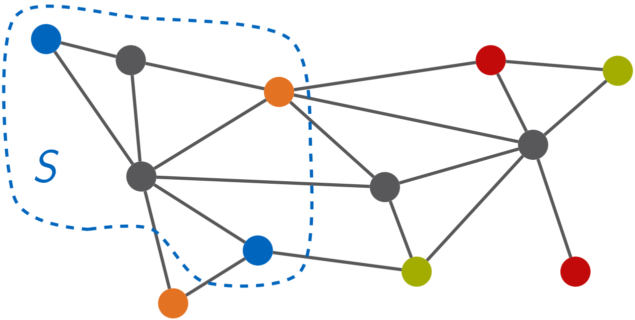
LP relaxation



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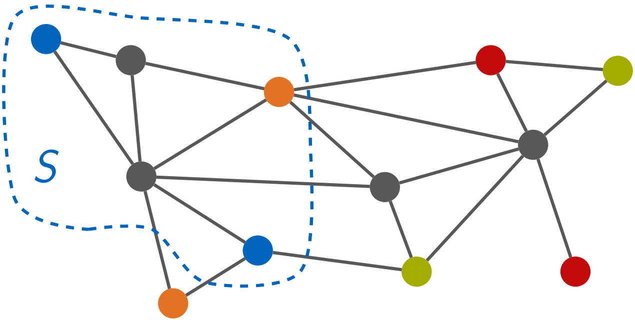
\text{s.t.} \quad & \sum_{e \in \delta(S)} x_e \;\geq\; f(S) \quad && \forall\, S \subseteq V \\
& 1 \;\geq\; x_e \;\geq\; 0 && \forall\, e \in E
\end{aligned}$$

LP relaxation



$$f(S) := \max\{r_{vw} \;:\; v \in S,\, w \in V \setminus S\}$$

$$[\mathrm{LP}(F)] \quad \min \sum_{e \in E \setminus F} w_e x_e$$

$$\text{s.t.} \quad \sum_{e \in \delta(S) \setminus F} x_e \;\geq\; f(S) - |\delta(S) \cap F| \qquad \forall\, S \subseteq V$$

$$1 \;\geq\; x_e \;\geq\; 0 \qquad \forall\, e \in E \setminus F$$

Algorithm

$$
\begin{aligned}
[\mathrm{LP}(F)] \quad & \min \sum_{e \in E \setminus F} w_e x_e \\
& \text{s.t.} \quad \sum_{e \in \delta(S) \setminus F} x_e \ \geq\ f(S) - |\delta(S) \cap F| \qquad \forall\, S \subseteq V \\
& \qquad\qquad 1 \ \geq\ x_e \ \geq\ 0 \qquad\qquad \forall\, e \in E \setminus F
\end{aligned}
$$

Algorithm (Iterated Rounding)

1. $F := \emptyset$
2. while (F is not feasible)
   - Compute basic optimal solution x to LP(F).
   - $F := F \cup \{e \in E \setminus F \ : \ x_e \geq 1/2\}$
3. return F

Remark: If LP is infeasible, there is no feasible solution to SND.

Main theorem

$$
\begin{array}{llrcll}
[\mathrm{LP}(F)] & \min & \displaystyle\sum_{e \in E \setminus F} w_e x_e & & & \\
 & \text{s.t.} & \displaystyle\sum_{e \in \delta(S) \setminus F} x_e & \geq & f(S) - |\delta(S) \cap F| & \forall\, S \subseteq V \\
 & & 1 \;\geq\; x_e & \geq & 0 & \forall\, e \in E \setminus F
\end{array}
$$

Theorem 11.1

Let $F \subseteq E$ and x be a basic feasible solution to LP(F). Then F is feasible or there is $e \in E \setminus F$ with $x_e \geq 1/2$.

Main theorem

$$
\begin{array}{llrcll}
[\mathrm{LP}(F)] & \min & \displaystyle\sum_{e \in E \setminus F} w_e x_e & & & \\
& \text{s.t.} & \displaystyle\sum_{e \in \delta(S) \setminus F} x_e & \geq & f(S) - |\delta(S) \cap F| & \forall\, S \subseteq V \\
& & 1 \;\geq\; x_e & \geq & 0 & \forall\, e \in E \setminus F
\end{array}
$$

Theorem 11.1

Let $F \subseteq E$ and x be a basic feasible solution to $\mathrm{LP}(F)$. Then F is feasible or there is $e \in E \setminus F$ with $x_e \geq 1/2$.

Theorem 11.2

There is a laminar collection $\mathcal{L} \subseteq 2^V$ such that

(1) $\sum_{e \in \bar{\delta}(S)} x_e = f'(S)$ for all $S \in \mathcal{L}$,

(2) $\{\chi_{\bar{\delta}(S)} : S \in \mathcal{L}\}$ is linearly independent,

(3) $\mathcal{L} = |\bar{E}|$.