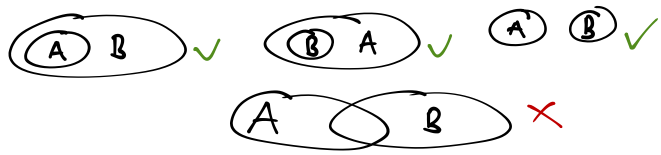
Survivable Network Design

Define $f'(S) := f(S) - |\delta(S) \cap F|$, $\bar{E} := \{e \in E \setminus F : x_e > 0\}$
and $\bar{\delta}(S) := \delta(S) \cap \bar{E}$ for $S \subseteq V$.

Definition Let $\mathcal{L} \subseteq 2^V$. $\mathcal{L}$ is laminar if for all $A, B \in \mathcal{L}$

$A \cap B = \emptyset$ or $A \subseteq B$ or $B \subseteq A$.



Proof of Lemma 11.4 (Sketch):

- Because x is basic feasible solution, there is $\mathcal{L} \subseteq 2^V$ fulfilling (1) - (3).
- Show that f' is weakly supermodular.
- By weak supermodularity: A, B fulfill (1) $\Rightarrow$ $A \cup B$, $A \cap B$ fulfill (1) or $A \setminus B$, $B \setminus A$ fulfill (1)

Exercise 6.2: Construct $\mathcal{L}$ so that it is laminar.

Proof of Theorem 11.3:

By contradiction assume $x_e < \frac{1}{2}$ for all $e \in \bar{E}$. For $S \in \mathcal{L}$ define

$$E_S := \{e = \{v,w\} \in \bar{E} : S \text{ is smallest set in } \mathcal{L} \text{ with } \{v,w\} \subseteq S\},$$

$$V_S := \{v \in V : S \text{ is smallest set in } \mathcal{L} \text{ with } v \in S\},$$

$$\phi(S) := \sum_{e \in E_S} 1 - 2x_e + \sum_{v \in V_S} \sum_{e \in \bar{\delta}(v)} x_e. \qquad (\text{"charging scheme"})$$

Claim 1 $\sum_{S \in \mathcal{L}} \phi(S) < |\bar{E}|$ **Claim 2** $\phi(S) \geq 1 \quad \forall S \in \mathcal{L}$

Claim 1 & 2 $\Rightarrow$ $|\bar{E}| > \sum_{S \in \mathcal{L}} \phi(S) \geq |\bar{E}|$ ↯

Proof of Claim 1:

($\mathcal{L} \neq \emptyset$ because F not feasible.)
Let $S' \in \mathcal{L}$ be $\subseteq$-max. Note that $\bar{\delta}(S') \neq \emptyset$ by (2) and let $e' \in \bar{\delta}(S')$.
Because S' is $\subseteq$-max in $\mathcal{L}$, $e' \notin E_S$ for all $S \in \mathcal{L}$.
Observe that $E_S \cap E_{S'} = \emptyset$ and $V_S \cap V_{S'} = \emptyset$ for $S \neq S'$. Thus:

$$\sum_{S \in \mathcal{L}} \phi(S) \leq \sum_{e \in \bar{E}} 1 - 2x_e + \underbrace{\sum_{v \in V} \sum_{e \in \bar{\delta}(v)} x_e}_{= \sum_{e \in \bar{E}} 2x_e} - (1 - 2x_{e'}) \leq |\bar{E}| - \underbrace{(1 - 2x_{e'})}_{> 0} < |\bar{E}|. \blacksquare$$

Proof of Claim 2: Let $S \in \mathcal{L}$. Let $\mathcal{A} := \{C \in \mathcal{L} : C \subseteq S\}$, "children" of S

Define edge sets: $\mathcal{C} := \{C \in \mathcal{A} : C \text{ is } \subseteq\text{-maximal in } \mathcal{A}\}$.

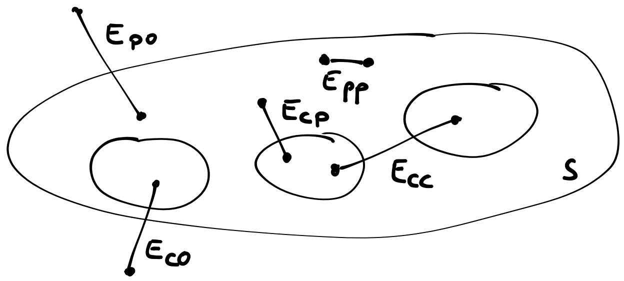
$$E_{cc} := \{e = \{v,w\} \in \bar{E} : v \in C,\ w \in C' \text{ for } C, C' \in \mathcal{C},\ C \neq C'\} \quad \text{"child-child"}$$

$$E_{cp} := \{e = \{v,w\} \in \bar{E} : v \in C',\ w \in S \setminus \bigcup_{C \in \mathcal{C}} C \text{ for } C' \in \mathcal{C}\} \quad \text{"child parent"}$$

$$E_{co} := \{e = \{v,w\} \in \bar{E} : v \in C,\ w \in V \setminus S \text{ for } C \in \mathcal{C}\} \quad \text{"child-out"}$$

$$E_{po} := \{e = \{v,w\} \in \bar{E} : v \in S \setminus \bigcup_{C \in \mathcal{C}} C,\ w \in V \setminus S\} \quad \text{"parent-out"}$$

$$E_{pp} := \{e \in \{v,w\} \in \bar{E} : v, w \in S \setminus \bigcup_{C \in \mathcal{C}} C\} \quad \text{"parent-parent"}$$



$$\begin{aligned}
\phi(S) &= \sum_{e \in E_{pp}} 1 + \sum_{e \in E_{cc}} 1 - 2x_e + \sum_{e \in E_{cp}} 1 - x_e + \sum_{e \in E_{po}} x_e \\
&= |E_{pp}| + |E_{cc}| + |E_{cp}| - 2x(E_{cc}) - x(E_{cp}) + x(E_{po}) \\
&= |E_{pp}| + |E_{cc}| + |E_{cp}| + x(\bar{\delta}(S)) - \sum_{C \in \mathcal{C}} \bar{\delta}(C) \\
&= |E_{pp}| + |E_{cc}| + |E_{cp}| + f'(S) - \sum_{C \in \mathcal{C}} f'(C)
\end{aligned}$$

$$\Rightarrow \phi(S) \in \mathbb{Z}$$

If $E_{cc} = E_{cp} = E_{po} = \emptyset$ then $\chi_{\bar{\delta}(S)} = \sum_{C \in \mathcal{C}} \chi_{\bar{\delta}(C)}$, contradicting (2).

Hence at least one of these sets is non-empty. $\Rightarrow \phi(S) > 0$

$\phi(S) \in \mathbb{Z}$ and $\phi(S) > 0 \Rightarrow \phi(S) \geq 1$ ∎