Iterated Randomized Rounding for STEINER TREE

$C_i :=$ component selected in iteration i $\quad F_i := \bigcup_{j=1}^{i-1} E_{C_j}$, $\quad \mathcal{C}_i :=$ dir. full comp. in G/F_i, $\quad x_i :=$ opt sol. to LP in iteration i

Lemma 13.1 $\mathbb{E}[d(T')] \leq 2\exp(-\ell/\Sigma)\cdot OPT$

Proof: $\mathbb{E}[mst(F_i \cup C_i)] = \mathbb{E}[mst(F_i) - drop_{F_i}(C_i)]$

$$= \mathbb{E}\Big[mst(F_i) - \tfrac{1}{\Sigma}\underbrace{\sum_{C\in\mathcal{C}_i} x_{i,C}\, drop_{F_i}(C)}_{\geq\, mst(F_i)}\Big]$$

$$\leq (1 - 1/\Sigma)\,\mathbb{E}[mst(F_i)]$$

$$\Rightarrow \mathbb{E}[T'] \leq (1-1/\Sigma)^{\ell}\cdot \underbrace{mst(\emptyset)}_{\leq 2OPT} \leq 2\exp(-\ell/\Sigma)\cdot OPT \quad \square$$

Lemma 13.2 $\mathbb{E}[d(E_{C_i})] \leq 1/\Sigma \cdot OPT$

Proof: $\mathbb{E}[d(E_{C_i})] = \mathbb{E}\Big[1/\Sigma \sum_{C\in\mathcal{C}_i} x_{i,C}\, d_{C_i}\Big] \leq \frac{1}{\Sigma}\cdot OPT \quad \square$

Contracting edges does not increase cost of LP.

because we do not solve LP exactly

Theorem 12.3 Iterated Rounding is a $(1+\ln(2)+\varepsilon)$-approximation for STEINER TREE.

Proof: $\mathbb{E}[ALG] = \mathbb{E}[d(T') + \sum_{i=1}^{\ell} d(E_{C_i})] \leq 2\exp(-\ell/\Sigma)\cdot OPT + \ell/\Sigma\cdot OPT$

$$= (1+\ln 2)\,OPT \leq 1.694\,OPT \quad \square$$

Choose $\Sigma := \frac{\lceil \ln 2 \cdot M\rceil}{\ln 2}$ and $\ell := \ln 2\cdot\Sigma$.

(M upper bound on $\sum_{C\in\mathcal{C}_i} x_C$, polynomial when restricting to components of constant size)

Proof of Drop Lemma (Idea): Let T be a MST in G/F. Use full comp. to construct helper graph $H=(R,\bar{E})$. For $e=\{v,w\}\in\bar{E}$, let $\bar{d}(e) := \max_{e\in T[v,w]} d(e)$ for $\bar{e}=\{v,w\}\in\bar{E}$ represent the drop from merging v and w.
Show that x induces a solution to bidirected cut relaxation.

$$\Rightarrow \sum_{C\in\mathcal{C}_i} drop_C\, x_C \geq \text{cost of MST in } H \text{ wrt } \bar{E} \geq d(T).$$

Semi-Definite Programming for MAX CUT

Lemma 13.4: Let $x \in \{-1, 1\}^n$ and $S := \{i \in V : x_i = 1\}$.

Then $\sum_{e \in \delta(S)} w(e) = \frac{1}{2} \cdot \sum_{\{i,j\} \in E} w_{ij} (1 - x_i x_j)$.

Proof: $\{i,j\} \in \delta(S) \iff i \in S, j \notin S$ or $i \notin S, j \in S \iff x_i \neq x_j$

Also: $1 - x_i x_j = \begin{cases} 2 & \text{if } x_i \neq x_j \\ 0 & \text{if } x_i = x_j \end{cases}$ for $x_i, x_j \in \{-1, 1\}$.

Thus: $\sum_{\{i,j\}} (1 - x_i x_j) = 2 \sum_{\{i,j\} \in \delta(S)} w_{ij}$ $\square$

Lemma 13.5 $OPT \leq Z^*$

Proof: Let $x^* \in \{-1, +1\}^n$ be an optimal solution to the QP.

Define $v_i = \begin{pmatrix} x_i^* \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then $v_i^T v_i = 1 \ \forall i \in [n]$ and

$$Z^* \geq \frac{1}{2} \sum_{\{i,j\} \in E} w_{ij} (1 - v_i^T v_j) = \frac{1}{2} \sum_{\{i,j\} \in E} (1 - x_i^* x_j^*) \underset{\text{Lemma 13.4}}{\geq} OPT. \quad \square$$

Proof of Theorem 13.6:

Consider $U := \text{span}(\{v_i^*, v_j^*\})$. Let r' be the projection of r on U.

Then $r'' := \frac{r'}{\|r'\|}$ is uniformly distributed on unit circle in U.

$$\mathbb{E}[ALG] = \sum_{\{i,j\} \in E} w_{ij} \Pr[\{i,j\} \in \delta(S)] = \sum_{\{i,j\} \in E} \frac{\arccos(v_i^{*T} v_j^*)}{\pi}$$

$$\geq 0.878 \sum_{\{i,j\} \in E} w_{ij} \frac{1}{2} (1 - v_i^{*T} v_j^*) = 0.878 \cdot Z^*. \quad \square$$

Solving the SDP: Replace $v_i^T v_j$ by a_{ij}.

Then $(a_{ij})_{i,j \in [n]}$ corresponds to sol. v_i, v_j if and only if the matrix $A := \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ is positive semidefinite.

Separation Oracle: Find smallest eigenvalue of A.

(see book for details)